PH 5840 Lecture - 7 17/8/2015

Note Title 8/13/2015

① Qubit density operator :-

Recall, $\left\{ \frac{I}{2}, \frac{\sigma_x}{2}, \frac{\sigma_y}{2}, \frac{\sigma_z}{2} \right\}$ form an ON basis for $M_2(\mathbb{C})$.

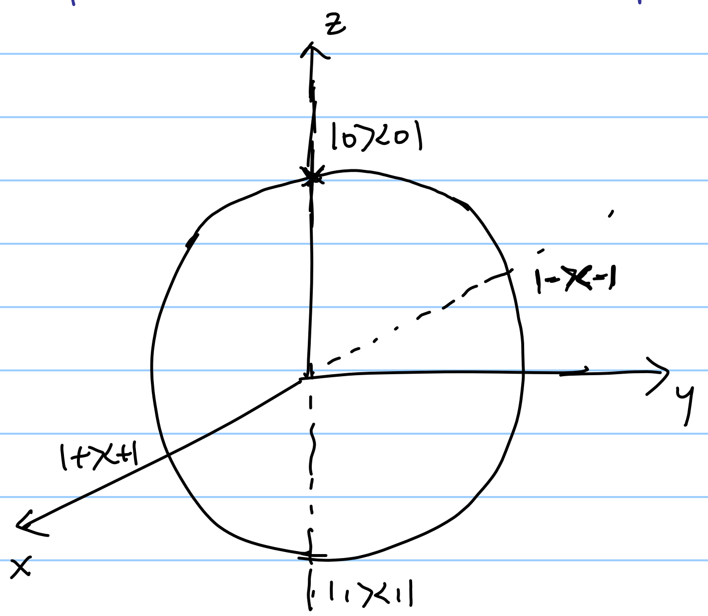
$$\therefore \rho = \frac{1}{2}\left(r_0 I + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z \right)$$

$Tr(\rho) = 1 \Rightarrow r_0 = 1$

$\rho^{+} = \rho \Rightarrow (r_x, r_y, r_z) \equiv \vec{r} \in \mathbb{R}^3$.

$Tr(\rho^2) \leq 1 \Rightarrow |\vec{r}| \leq 1$

Unit sphere in 3-d $\Rightarrow$ Bloch sphere!!



Origin: $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

$= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$

$\Rightarrow$ Maximally mixed state!

② Composite Systems & Entanglement

$$|\psi_A\rangle \in \mathcal{H}_A \, , \quad |\psi_B\rangle \in \mathcal{H}_B \, .$$

$\mathcal{H}_A$, $\mathcal{H}_B$ are called subsystems of $\mathcal{H}_{AB}$

$$|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \, .$$

$\otimes$ ↓ Kronecker product

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ \vdots & & & \\ a_{n1} B & a_{n2} B & \cdots & a_{nn} B \end{pmatrix}$$

Eg. Basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$:-

$$\{|0\rangle, |1\rangle\}_A \otimes \{|0\rangle, |1\rangle\}_B \, .$$

$$\equiv \{|0\rangle \otimes |0\rangle, \; |0\rangle \otimes |1\rangle, \; |1\rangle \otimes |0\rangle, \; |1\rangle \otimes |1\rangle\}$$

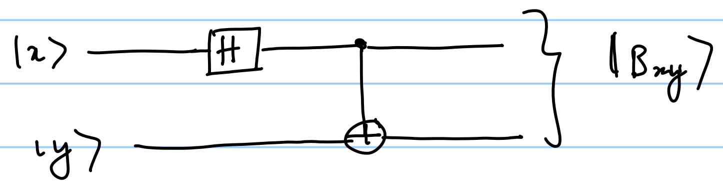
* Rules:

* $\langle v_A \otimes w_B, e_A \otimes f_B \rangle = \langle v_A, e_A \rangle \langle w_B, f_B \rangle$

* $(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$

* Bell states:-

Consider this circuit:-



$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \, , \quad |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \, , \quad |\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

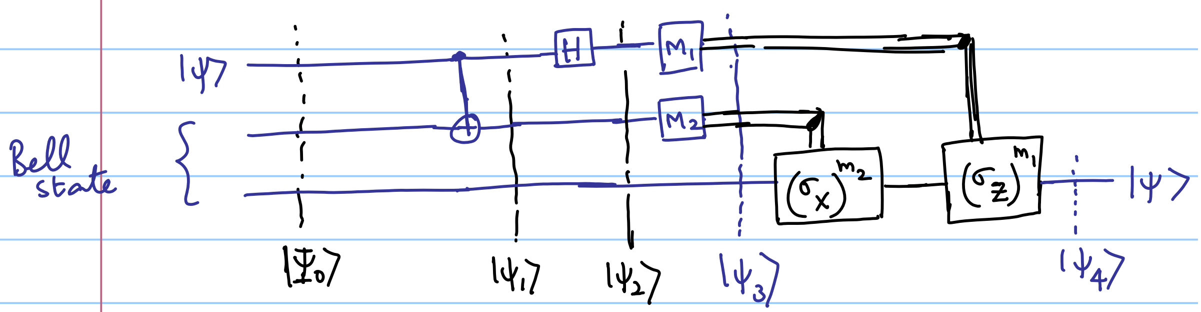
$\{|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle\}$ forms an ON basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$.

* There do not exist states $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^2$ such that

$$|\beta_{xy}\rangle = |\alpha\rangle \otimes |\beta\rangle$$

$\Rightarrow$ These are not product states; entangled!

(2a) Teleportation:- Entangled states are an important resource for quantum information and computing.



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ ("unknown" quantum state)

- Single copy of $|\psi\rangle$ given to Anita.
- Her task is to communicate this state to Bharat without destroying the state! (i.e. without measuring the state!)
- Resource: Anita & Bharat share a Bell state.

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \in \mathcal{H}_{A_1}$$

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \in \mathcal{H}_{A_2} \otimes \mathcal{H}_B$$

- $|\psi_0\rangle_{A_1A_2B} = \left(\alpha|0\rangle+\beta|1\rangle\right)_{A_1} \otimes \frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)_{A_2B}$
$$= \frac{1}{\sqrt{2}}\left[\alpha|0\rangle\left(|00\rangle+|11\rangle\right)+\beta|1\rangle\left(|00\rangle+|11\rangle\right)\right]$$

- $|\psi_1\rangle_{A_1A_2B} = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle\left(|00\rangle+|11\rangle\right)+\beta|1\rangle\left(|10\rangle+|01\rangle\right)\right]$

- $|\psi_2\rangle_{A_1A_2B} = \frac{1}{2}\left[\alpha\left(|0\rangle+|1\rangle\right)\left(|00\rangle+|11\rangle\right)+\beta\left(|0\rangle-|1\rangle\right)\left(|10\rangle+|01\rangle\right)\right]$

$$= \frac{1}{2}\Big[|00\rangle_A\left(\alpha|0\rangle+\beta|1\rangle\right)_B + |11\rangle_A\left(\alpha|1\rangle-\beta|0\rangle\right)_B$$
$$+|01\rangle_A\left(\alpha|1\rangle+\beta|0\rangle\right)_B + |10\rangle_A\left(\alpha|0\rangle-\beta|1\rangle\right)_B\Big]$$

- Anita measures her qubits in $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$ basis.

(m_1, m_2)	B's state
$(0,0)$	$\alpha|0\rangle+\beta|1\rangle = |\psi\rangle$!
$(0,1)$	$\alpha|1\rangle+\beta|0\rangle = \sigma_x|\psi\rangle = (\sigma_x)^{m_2}(\sigma_z)^{m_1}|\psi\rangle$
$(1,0)$	$\alpha|0\rangle-\beta|1\rangle = \sigma_z|\psi\rangle = (\sigma_x)^{m_2}(\sigma_z)^{m_1}|\psi\rangle$
$(1,1)$	$\alpha|1\rangle-\beta|0\rangle = \sigma_x\sigma_z|\psi\rangle = (\sigma_x)^{m_2}(\sigma_z)^{m_1}|\psi\rangle$

Note: No faster than light communication!
* B gets $|\psi\rangle$ only after A has communicated her measurement outcomes.
* Without the classical channel, teleportation conveys No information.

History:

Quantum teleportation discovered by Bennett, Brassard, Crepeau, Jozsa, Peres and Wootters, PRL 70, 1895 (1993).

- Experimentally:- Bouwmeester et al (1997) (photon polarization)
  Furasawa et al (1998) ("squeezed" states of light)
  Nielsen et al (1998) (NMR)

(2b) Reduced density operator : Partial trace

Given $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$,

$$\rho_B = \mathrm{Tr}_A[\rho_{AB}) = \sum_i \langle i_A | \rho_{AB} | i_A \rangle,$$

where, $\{|i_A\rangle\}$ is an ON basis for $\mathcal{H}_A$.

||ly $\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_j \langle j_B | \rho_{AB} | j_B \rangle$,

where, $\{|j_B\rangle\}$ is an ON basis for $\mathcal{H}_B$.

Examples: (Ia) $\rho_{AB} = |a\rangle\langle a| \otimes |b\rangle\langle b| = |ab\rangle\langle ab|$

$$\rho_A = (\langle b|b\rangle)\, |a\rangle\langle a| \;, \quad \rho_B = (\langle a|a\rangle)\, |b\rangle\langle b|$$

(Ib) $\rho_{AB} = \rho_A \otimes \sigma_B$, where,

ρ_A is a density operator on $\mathcal{H}_A$

ρ_B is a density operator on $\mathcal{H}_B$.

$$\Rightarrow \mathrm{Tr}_A(\rho_{AB}) = \mathrm{Tr}_A(\rho_A)\,\sigma_B = \sigma_B$$

$$\mathrm{Tr}_B(\rho_{AB}) = \mathrm{Tr}_B(\sigma_B)\,\rho_A = \rho_A.$$

(II) $|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$$\rho_{AB} = \frac{1}{2}\Big(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\Big)$$

$$\mathrm{Tr}_A(\rho_{AB}) = \mathrm{Tr}_B(\rho_{AB}) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$= \frac{I}{2} \;//$$

(2c) Teleportation : a local point-of-view!

$$|\psi_2\rangle_{A_1A_2B} = \frac{1}{2}\Big[|00\rangle_A(\alpha|0\rangle+\beta|1\rangle)_B + |11\rangle_A(\alpha|1\rangle-\beta|0\rangle)_B$$
$$+|01\rangle_A(\alpha|1\rangle+\beta|0\rangle)_B + |10\rangle_A(\alpha|0\rangle-\beta|1\rangle)_B\Big]$$

$\alpha|0\rangle+\beta|1\rangle = |\phi_0\rangle_B$	$\alpha|0\rangle-\beta|1\rangle = |\phi_2\rangle_B$
$\alpha|1\rangle+\beta|0\rangle = |\phi_1\rangle_B$	$\alpha|1\rangle-\beta|0\rangle = |\phi_3\rangle_B$

$\therefore$ Density operator after Anita's measurement,
(post-measurement state)

$$\rho_{A_1A_2B} = \frac{1}{4}\Big[|00\rangle\langle 00|_A \otimes |\phi_0\rangle\langle\phi_0|_B + |01\rangle\langle 01| \otimes |\phi_1\rangle\langle\phi_1|_B$$
$$+ |10\rangle\langle 10| \otimes |\phi_2\rangle\langle\phi_2|_B$$
$$+ |11\rangle\langle 11|_A \otimes |\phi_3\rangle\langle\phi_3|_B\Big]$$

$\therefore$ Reduced state on Bharat's side:

$$\rho_B = \frac{1}{4}\sum_{i=0}^{3} |\phi_i\rangle\langle\phi_i|_B$$

$$= \frac{2(|\alpha|^2+|\beta|^2)|0\rangle\langle 0| + 2(|\alpha|^2+|\beta|^2)|1\rangle\langle 1|}{2}$$

$$= \frac{I}{2}\;!$$

$\therefore$ Without classical communication,
ρ_B is completely independent of $|\psi\rangle$
$\Rightarrow$ B has no information about $|\psi\rangle$.